

Probability distributions for multicomponent systems with multiplicative noise.

J.M. Deutsch

University of California

Santa Cruz CA 95064

Abstract

Linear systems with many degrees of freedom containing multiplicative and additive noise are considered. The steady state probability distribution for equations of this kind is examined. With multiplicative white noise it is shown that under some symmetry conditions, the probability distribution of a single component has power law tails, with the exponent independent of the strength of additive noise, but dependent on the strength of the multiplicative noise. The classification of these systems into two regimes appears to be possible in the same manner as with just one degree of freedom. A physical system, that of a turbulent fluid undergoing a chemical reaction is predicted to show a transition from exponential to power law tails, as the reaction rate is increased. A variety of systems are studied numerically. A replication algorithm is used to obtain the Lyapunov exponents for high moments, which would be inaccessible by more conventional approaches.

I. INTRODUCTION

This paper analyzes coupled linear equations containing additive and multiplicative noise. They can be written in the form

$$\dot{\vec{\phi}} = \mathbf{M} \cdot \vec{\phi} + \mathbf{A} \cdot \vec{\phi} + \vec{\eta} \quad (1)$$

$\vec{\phi} \equiv (\phi_1, \phi_2, \dots, \phi_N)$ represents the variables of interest $\mathbf{M}(t)$ is a matrix that varies randomly with time. \mathbf{A} is a time independent matrix and $\vec{\eta}$ is a random time dependent vector. To motivate the study of eqn. (1) we will illustrate a number of physical problems that are in this category.

The motion of dye, or a temperature field, in a randomly stirred fluid leads to an equation for the dye density where the random velocity \mathbf{v} is coupled to dye density ϕ in a multiplicative way $\mathbf{v} \cdot \nabla \phi$. obeying the equation

$$\partial_t \phi + \mathbf{v} \cdot \nabla \phi = d \nabla^2 \phi \quad (2)$$

In two and higher dimensions this is normally studied with the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. d is a diffusion constant, and $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ is a random function of position and time. This problem has received much recent attention [1–11].

The Schroedinger equation with a random time dependent potential is another example [12,13], although it will not be investigated here, and this is almost identical to the equation for wave propagation in random media [14,15]. Population growth models which are relevant to chemical reactions [16], population biology [17] and low temperature quantum systems [18,19] are often of this form

$$\partial_t \phi = \alpha \phi + f \phi + d \nabla^2 \phi \quad (3)$$

where f is a random function of position and time, α and d are parameters. Dye laser theory also leads to an equation with both types of noise [20]. Polymers in turbulent flow also make use of such equations [21,22].

The inclusion of additive noise most frequently corresponds to adding thermal fluctuations to the problem. In some cases it is a source term and replaces the effect of a boundary. For example in turbulent Raleigh-Bernard convection, where a liquid is heated from below,

hot “plumes” [23] form and rise up through a liquid. In this case ϕ in eqn. (2) represents the temperature field, and a source term in this equation should be added representing heat flow through the boundaries.

A one component problem similar to that studied here has been analyzed by Drummond [24]. However his analysis is not completely applicable to the systems mentioned above as in these cases the additive and multiplicative noise are uncorrelated with each other. However he also finds two regimes, one with all moments defined and the other with a divergence at a finite moment.

The classification into different two regimes is very similar to the one component case [25] where it was shown that there are two different types of behavior found for the probability distribution function (PDF). There it was shown that the PDF has power law tails in one regime and stretched exponential in another. In section III the n-component vector version of these equations is considered for the case where both kinds of noise are white and Gaussian. Here under certain symmetry conditions it is shown that one also finds a power law tail for the PDF of any of the vector’s components.

An important example of non-power law tails involving many components is eqn. (2). $L(q)$ in this case should lead to a stretched exponential or an exponential tail for the PDF of dye density. Such distributions have been observed experimentally and have already been the subject of much theoretical work. The argument presented here for these tails is much general and just relies on mass conservation, that fact that dye is conserved.

Is there a physical system similar to eqn. (2) where the dye particles are not conserved? An example of such a system would be a reactive fluid, where heat is generated by particles reacting with each other. Such a system would have Lyapunov exponents that cross over to the first regime. This leads to the interesting prediction that such systems can exhibit power tails. This will be discussed in section IV. Numerical confirmation of this prediction is given in section V.

II. DISTRIBUTION IN THE ABSENCE OF ADDITIVE NOISE

Here we will consider eqn. (1) when the additive noise term $\vec{\eta}$ is zero. In this case there is not a well defined steady state probability distribution. However this case can also be

understood by considering the time dependence of the distribution for long times. We shall see that there is a scaling form of this distribution for long times which is identical in form to the one-component case.

A. Scaling of the probability distribution

Without the additive noise, the formal solution to eqn. (1) is

$$\vec{\phi}(t) = T e^{\int_0^t \mathbf{M}(\tau) + \mathbf{A} d\tau} \vec{\phi}(0) \quad (4)$$

Here T denotes the time ordered product. If this integration is discretized into time intervals Δt , then this is equivalent to the problem of the multiplication of $t/\Delta t$ random matrices of the form $\exp(\Delta t(\mathbf{M}_i + \mathbf{A}))$. The matrices $M_i = M(i\Delta t)$ are random and independent because we are considering $M(t)$ to have a white noise spectrum.

The multiplication of random matrices has been well studied. For long times it has been proved that $|\vec{\phi}(t)|$ has an exponential dependence on time for long times [26]. Fluctuations in this system are large in the sense that higher moments do not scale in a trivial way with lower moments. In one considers a single component of ϕ , ϕ_1 then if $\phi_1 \sim \exp(\gamma_1 t)$, then $\phi_1^2 \sim \exp(\gamma_2 t)$, where γ_1 and γ_2 are not trivially related. This can be seen to be true even in the scalar case where the matrices are dimensions 1×1 and hence commute. In summary, one can define a generalized Lyapunov exponent $L(q)$ by [27,28]

$$\langle \phi_i^q \rangle \propto e^{L(q)t} \quad (5)$$

where the brackets denote an ensemble average over the noise \mathbf{M} .

Now we would like to work out the form of the PDF of ϕ , one of the components of $\vec{\phi}$. This situation is similar to the problem of multifractals and its solution works out the same way. It can be checked that the distribution giving such scaling is

$$\ln P(\ln \phi) \propto t f((\ln \phi)/t) + \mathcal{O}\left(\frac{\ln t}{t}\right) \quad (6)$$

Another way of deriving this scaling form is by a thermodynamic analogy [27,28]. The discretized version of eqn. (4) can be thought of as a coupled one dimensional spin system,

where each spin has N states but with random interactions \mathbf{M}_i . Eqn. (4) is then the partition function of the system for a particular realization of the \mathbf{M}_i .

From eqn. (6) it is clear that $P(\phi)$ does not tend towards a time independent, that is steady state, distribution. $\langle (\ln \phi)^2 \rangle \propto t$ for large t , so the the distribution continues to increase for small and large $|\phi|$.

III. PROPERTIES FOR MANY COMPONENTS

A. Equation for moments

We will start by considering the coupled differential eqns. (1) where \mathbf{M} is a random gaussian matrix with $\langle M_{ij}(t)M_{kl}(t') \rangle = 2\Gamma_{ijkl}\delta(t-t')$, and where \mathbf{A} is taken to be symmetric, real, and time independent. As will be seen below, the elements of \mathbf{A} must be sufficiently negative so that there is a well defined stationary solution, that is the solution does not diverge for long times. $\vec{\eta}$ is additive random noise that is taken to be white and uncorrelated $\langle \eta_i(t)\eta_j(t') \rangle = D\delta(t-t')\delta_{i,j}$. Later in this section we will see that when $\mathbf{M}(t)$ is either real symmetric or anti-symmetric, the steady state probability distribution for ϕ_i has a power law tail.

Using standard methods [29] it is possible to derive the steady state probability distribution for $\vec{\phi}$.

$$\left[\frac{\partial}{\partial t} + \sum_i^N \frac{\partial}{\partial \phi_i} \left(\sum_j A_{ij} \phi_j + \sum_{jl} \Gamma_{ijjl} \phi_l \right) - \sum_{ik} \frac{\partial^2}{\partial \phi_i \partial \phi_k} \sum_{jl} \Gamma_{ijkl} \phi_j \phi_l - \sum_i D \frac{\partial^2}{\partial \phi_i^2} \right] P(\vec{\phi}) = 0 \quad (7)$$

Next consider n th moments of this equation. Multiply through this equation by $\phi_{\alpha_1} \phi_{\alpha_2} \dots \phi_{\alpha_n}$ where the α_i 's can take on any integral value between 1 and N . Integrating this over all the ϕ_i 's gives an equation relating moments of order n to moments of order $n-2$. It is convenient to write such an equation in matrix form. Define $\vec{\alpha} \equiv (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\vec{\beta} \equiv (\beta_1, \beta_2, \dots, \beta_n)$ where the α_i 's and β_i 's can take on any integral value between 1 and N . Define $\vec{\phi}_{\vec{\alpha}} \equiv \langle \phi_{\alpha_1} \phi_{\alpha_2} \dots \phi_{\alpha_n} \rangle$. Then the equation for the n th moment can be written in the form

$$\sum_{\vec{\beta}} (\mathbf{a}_{\vec{\alpha}, \vec{\beta}} - \mathbf{G}_{\vec{\alpha}, \vec{\beta}}) \vec{\phi}_{\vec{\beta}} = N_{\vec{\alpha}} - \frac{\partial \vec{\phi}_{\vec{\alpha}}}{\partial t} \quad (8)$$

where

$$\mathbf{a}_{\vec{\alpha}, \vec{\beta}} = - \sum_{i=1}^n A_{\alpha_i, \beta_i} \prod_{\substack{j \\ j \neq i}}^n \delta_{\alpha_j, \beta_j}, \quad (9)$$

$$G_{\vec{\alpha}, \vec{\beta}} = \sum_{i=1}^n \sum_{\gamma=1}^N \Gamma_{\alpha_i \gamma \gamma \beta_i} \prod_{\substack{j \\ j \neq i}}^n \delta_{\alpha_j, \beta_j} + \sum_{\substack{i, j \\ i < j}}^n [\Gamma_{\alpha_i \beta_i \alpha_j \beta_j} + \Gamma_{\alpha_i \beta_j \alpha_j \beta_i}] \prod_{\substack{k \\ k \neq i, j}}^n \delta_{\alpha_k, \beta_k}, \quad (10)$$

and

$$N_{\vec{\alpha}} = D \sum_{\substack{i, j \\ i < j}}^n \delta_{\alpha_i, \alpha_j} \phi_{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_N} \quad (11)$$

This involves moments of order $n - 2$.

Eqn. (8) provides a convenient framework to analyze the general properties of moments. It is of the form of a matrix equation of order N^n . Using this, it will be shown that these moments diverge at sufficiently high order if the symmetry condition, mentioned above, for \mathbf{M} holds and therefore the PDF of the ϕ 's has a power law tail.

When there is no additive noise, that is $\mathbf{N} = \mathbf{0}$, the Lyapunov exponent $L(n)$ is given by the largest eigenvalue of the matrix $\mathbf{G} - \mathbf{a}$.

B. Divergence of moments

With finite additive noise in steady state, Eqn. (8) can be rewritten

$$\vec{\phi}_{\vec{\alpha}} = - \sum_{\vec{\beta}} (\mathbf{G} - \mathbf{a})^{-1}_{\vec{\alpha}, \vec{\beta}} \mathbf{N}_{\vec{\beta}} \quad (12)$$

The objective here is to show that there exists some finite n beyond which the moments do not exist. Generically this will occur when the highest eigenvalue of $\mathbf{G} - \mathbf{a}$ passes through zero. Positive eigenvalues show that the solution has become ill defined, and that the moments have ceased to exist. Note that if the n th moments exist, the moments of order $n - 2$ must also exist, and hence the right hand side of eqn. (12) exists. Therefore one accomplishes the above objective by showing that the inverse matrix on the right hand side of eqn. (12) has eigenvalues that must pass from negative to positive, as n is increased.

From the end of section III A , the value of n where this occurs corresponds to the point where $L(n)$ passes through zero. This condition is in agreement with the result obtained through the heuristic argument of the preceding publication [25].

Thus to show this divergence, one must find the dependence of the eigenvalues of $\mathbf{G} - \mathbf{a}$ on n , the order of the moments. One must show that at least one of the eigenvalues of this matrix become positive for sufficiently large n .

Note that the eigenvalues of $\mathbf{G} - \mathbf{a}$ will span some finite domain E_{min} to E_{max} both of which are difficult to ascertain. However one can find a number between these two extremes by forming the scalar product $\hat{\mathbf{x}} \cdot (\mathbf{G} - \mathbf{a}) \cdot \hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is any unit vector. If this scalar product becomes positive, then at least one eigenvalue is greater than zero, and therefore the n th moment is divergent. If one makes the choice $\hat{\mathbf{x}} \propto (1, 1, \dots, 1)$ then the scalar product can be evaluated. At this point we restrict the random matrices \mathbf{M} to be either real symmetric or anti-symmetric.

$$\begin{aligned} \sum_{\vec{\alpha}, \vec{\beta}} x_{\alpha} (\mathbf{G}_{\vec{\alpha}, \vec{\beta}} - a_{\vec{\alpha}, \vec{\beta}}) x_{\beta} = \\ \frac{n}{N} \sum_{\gamma, \delta=1}^N A_{\gamma, \delta} + \frac{n}{N} \sum_{\gamma=1}^N \langle (\sum_{\alpha=1}^N M_{\alpha, \gamma})^2 \rangle + 2 \frac{n(n-1)}{N^2} \langle (\sum_{\alpha, \beta=1}^N M_{\alpha, \beta})^2 \rangle \end{aligned} \quad (13)$$

Now we need to analyze the behavior of this expression as a function of n . The first two summations have prefactors proportional to n , where as the last term is proportional to $n(n-1)$ and must be positive. Therefore this scalar product must become positive for sufficiently large n .

If we consider this scalar product as the A_{ij} 's are varied one obtains a similar result. Suppose we start with A_{ij} 's that are sufficiently negative that the n th moment exists. As the A_{ij} 's become less negative, at some point the scalar product becomes positive. Therefore at this point the moments $\vec{\phi}_{\vec{\alpha}}$ are ill defined. Therefore for a fixed order of moments n , the moments can be made ill defined by varying the matrix \mathbf{A} .

These arguments show that for a system described by eqn (1), there must always be moments of the ϕ 's that diverge if sufficiently high moments are examined. All of the moments of a given order are expected to diverge at the same time, if the matrix Γ is not transformable to block diagonal form. Block diagonal form should only occur when the system of eqns. (1), can be decoupled into completely separate subsystems. Aside from

this case, all elements of $(\mathbf{G} - \mathbf{a})^{-1}$ will become divergent at the same n . This means for example, that $\langle \phi_1^n \rangle$ should diverge at the same n as $\langle \phi_2^{n-3} \phi_3^3 \rangle$. This argument shows that the divergence in moments should take place somewhere between the n th and $(n+2)$ th moment. It is quite reasonable to assume that the power law tail for the PDF will take on a non-integral value. Note that the divergence is not dependent on the matrix \mathbf{N} , which is a function of D the strength of the additive noise. Therefore the exponent of the power law should also be unaffected by the strength of the additive noise. However the steady state distribution becomes ill defined in the limit of zero additive noise as we saw in section II. The power law tail should only depend on $\mathbf{\Gamma}$ and \mathbf{A} .

IV. PHYSICAL EXAMPLE

Here the results of the previous section are used to understand and make predictions about an interesting physical system, the convection of a passive scalar field such as temperature in a random velocity flow. We will add the extra ingredient that the fluid is undergoing an irreversible chemical reaction $A \rightarrow B$. All molecules start off as type A and the reaction is exothermic. In general one expects the rate constant to be temperature dependent. Over a limited temperature range this dependence can be approximated as linear. It is easily seen that this reaction adds a term $\alpha\phi$ to the right hand side of eqn. (2). The equation under consideration is

$$\partial_t \phi + \mathbf{v} \cdot \nabla \phi = d \nabla^2 \phi + \alpha \phi + \eta(x, t) \quad (14)$$

The last term is an additive random noise term, the correlation function in fourier space $\langle |\hat{\eta}(\mathbf{k}, t)|^2 \rangle$ must go to zero as $k \rightarrow 0$ to ensure heat conservation in the absence of chemical reactions.

It will now be argued that this system exhibits a cross-over from an exponential tail in the PDF, $P(T)$, to a power law as α is increased.

First consider the case of no reaction, $\alpha = 0$ and no additive noise. In this case we will now see that $L(q)$ must asymptote to a constant ≤ 0 . There are three ingredients in this argument. First the Laplacian acts as a short distance cutoff precluding variation of the temperature field on a smaller scale. To estimate this length scale we are assuming

that the velocity field has a small distance cutoff. In the case of turbulence this would be Kolmogorov's smallest eddy scale. Another scale of relevance is obtained by comparing the first two terms on the right hand side of eqn. (14). This yields a scale $l_c = d/v_{rms}$. As is usual with competition between convection and diffusion, it is the latter that will dominate below and the former that will dominate above the scale l_c [30]. Therefore at a sufficiently small length scale the variation of T should be smooth. Second, in a closed system or one with zero net heat flux into the system, the integrated temperature is conserved. Third if the scalar field is initially positive everywhere, it must remain so under evolution of eqn. (2). These three observations, of (a) conserved heat flux, (b) minimum length scale, and (c) positivity of ϕ , imply that in steady state there will be a maximum that the temperature can take. This would be when the entire non-zero, or non-average, temperature field is concentrated in a spike of size the minimum length scale. Because there is a maximum to ϕ , it cannot grow exponentially and therefore $L(q) \leq 0$.

But in fact from common experience one knows that without additive noise temperature variations will decrease in time. Physically it is apparent, that after some time an isolated system should come to equilibrium at a constant temperature. The form of the decrease in r.m.s. temperature fluctuations depends on the form of the random velocity field. The decrease in fluctuations will depend, for long times, on the boundary conditions. If there is no heat flux in or out of the system, the integrated temperature is conserved. With $\mathbf{v} = \mathbf{0}$ the process is purely diffusional, and one expects an exponential decay in fluctuations of the temperature field. The addition of a random velocity field should not change this conclusion. It should have the effect of mixing the system faster and increasing the rate of decay. If the boundary conditions are no longer insulating so that heat flows out of the system, the rate of decay should decrease further. The exact dependence is not important for the conclusions here, but only that in a finite size system and for a variety of boundary conditions, $L(q) < 0$ for $q > 0$.

Now consider the effect of adding a term $\alpha\phi$ to the right hand side of eqn. (2) to make the fluid reactive as described above. By letting $\phi \rightarrow \phi \exp(\alpha t)$, one eliminates the term $\alpha\phi$ from the equation. Therefore the term $\alpha\phi$ modifies the Lyapunov exponents with no reaction $L_0(q)$ to $L(q) = L_0(q) + \alpha q$. Because $L_0(q)$ must be convex, sufficiently large α

must cause $L_0(q) + \alpha q$ to cross zero at finite q .

If we now consider the steady state behavior of $P(\phi)$ with additive noise we recall that because $L(q) \leq 0$ for $\alpha = 0$, $P(\phi)$ for one component models [25] must be an exponential or stretched exponential. For sufficiently large α , one should observe a cross over to power law behavior. The precise cross over behavior cannot be predicted from this analysis, only that such a cross over must take place. The next section confirms this cross over numerically.

V. NUMERICAL SIMULATIONS

We will illustrate and confirm some of the results presented here by means of computer simulation. Consider eqn. (2) in one dimension. In one dimension the incompressibility constraint cannot be enforced and still obtain an interesting problem, so this constraint is dropped. The equation is discretized spatially giving

$$\dot{\phi}_i + (\phi_{i+1}v_{i+1} - \phi_i v_i) = d(\phi_{i+1} - 2\phi_i + \phi_{i-1}) \quad (15)$$

This equation is solved with periodic boundary conditions, and it is clear that the total ϕ $\Phi \equiv \sum \phi_i$ is conserved. Since we are interested in the fluctuations it is most convenient to take $\Phi = 0$. The simplest “order parameter” for this system is the total standard deviation.

$$\phi^2 \equiv \sum_{i=1}^N \phi_i^2 \quad (16)$$

This is to be distinguished from the ϕ ’s being evolved by eqn. (1). In this section we will denote the order parameter simply by ϕ . The different v_i ’s are taken to be independent and Gaussian distributed. This equation is updated by fourth order Runge Kutta. 16 lattice sites are used, and the distribution $P(\phi)$ was computed by averaging. Fig. 1 plots $(\ln P(\phi))/t$ versus $(\ln \phi)/t$ for two different times t . According to eqn. (6) the resulting function should be independent of time, up to an overall vertical shift. The two curves have been shifted relative to each other and lie on top of each other within the error bars confirming the validity of the scaling assumed in this paper. Note that the data for the longer time PDF does not extend down as far as for the shorter time. There is a problem measuring the tails of the PDF because the number of data points becomes very small. This one dimensional

problem may not have an $L(q)$ that is less than zero for positive q as the positivity of ϕ_i is no longer guaranteed.

We next turn to the case where additive noise is included. We shall illustrate the transition in behavior from an exponential tail in the PDF to a power law tail in the case of a reactive randomly stirred fluid. Adding a term $\alpha\phi_i$ to the right hand of eqn. (15) represents a one dimensional discretized version of the reacting system described in section IV. Periodic boundary conditions are still employed here but we start with the condition that the sum over all sites of ϕ_i is zero. In this case for a finite size system and $\alpha = 0$, $\langle\phi_i\rangle$ will decrease exponentially. This is seen from the fact that when the velocity field is zero, the system is diffusive and ϕ_i decreases exponentially. When velocity is added, this does not alter this result. From the previous section it was predicted that a system with sufficiently large α should have power law tails.

Fig. (2) shows ϕ versus $P(\phi)$ for different values of α . If α is too large all the $L(q)$ are positive and the system is unstable. For α too small there is no discernible power law behavior. However this cannot be ruled out based on the data. There is a narrow region where clear power law tails are manifest.

The method just used for obtaining the PDF has the draw back of not being able to probe the tails of the distribution. Equivalently, the high order moments and their $L(q)$'s are difficult to obtain. It is the convergence of high order moments that presents a problem. Multiplicative noise equations like eqn. (1) have highly intermittent behavior. This is a result of the fact that different moments have different Lyapunov exponents. Therefore the typical behavior of a high moment will be very different from its true average. Fluctuations that are exponentially unlikely in time dominate the correct answer. Therefore straightforward averaging is practically limited to very low q , before the number of number of runs that are needed to obtain convergence becomes far too large.

A way around this problem is to use a “replication algorithm”. Similar methods have been used in quantum simulations for the past fifteen years [31]. In order for this method to work, one needs the order parameter ϕ to be positive definite, which it is in most situations, as in the above example. A large number of copies of the system described by eqn. (1) are made. These are all updated to the next time step, with independently chosen random

numbers, the matrices \mathbf{M} . A weight w is computed by taking the ratio $w = \phi^n(t + \Delta t) / \phi^n(t)$. Again ϕ can be taken to be the total ϕ field, or the standard deviation of it, as above. Using this total rather than ϕ_i at a single point i leads to better convergence and is therefore what is done in practice. Once the weights w are determined, the integer part of w is taken and that number of copies is made. The remainder of w is taken into consideration, by generating a random number between 0 and 1 and making an additional copy if the random number is smaller than the remainder. Additionally, to stop the exponential explosion in the number of copies, the weights are all normalized so that the number of copies is kept almost constant.

The above method takes into account the idea of *importance sampling*. When one is interested in the average value of the moment $\phi^{n+1}(t)$, this is not computed directly, but the average of ϕ is computed. It is computed for a biased ensemble of configurations, each one having a probability proportional to ϕ^n of appearing. Therefore one is computing $\langle \phi^n \phi \rangle / \langle \phi^n \rangle$. This method is most powerful with no additive noise. In this case the exponential divergence of this ratio is computed for long times, which gives $L(n+1) - L(n)$.

The method is illustrated first for the two dimensional version of eqn. (2) with no additive noise. To make sure that $\nabla \cdot \mathbf{v} = 0$, \mathbf{v} is derived from a vector potential $\mathbf{v} = \nabla \times \mathbf{A}$. The velocity field \mathbf{v} is taken to be smooth but randomly varying in time. A is taken to be of the form

$$\mathbf{A}(x, y, t) = M(t) \sin\left(\frac{2\pi(x - x_0(t))}{L}\right) \sin\left(\frac{2\pi(y - y_0(t))}{L}\right) \hat{z} \quad (17)$$

New values of x_0 and y_0 are randomly chosen every Δt steps. $M(t)$ is a Gaussian random variable. This describes a circular motion with periodic boundary conditions, moving with a random angular velocity. There is no problem obtaining satisfactory convergence even for large moments. The value of $\ln \phi$ as a function of time is plotted for three different values of q in fig. 3 for an 8×8 system.. Approximately 720 copies are run in this simulation. The slopes of these lines give the difference in Lyapunov exponents $\Delta L \equiv L(q+1) - L(q)$. This difference is plotted versus q in fig. 4 in a system of size 8×8 . The same quantities are plotted in fig. 5 in a system of size 16×16 . All other parameters are kept the same and 180 copies were updated. Note that the variation of $L(q)$ is much smaller for this larger system. Also note that $L(q)$ is always negative for positive q and appears convex. By adding

a chemical reaction of the type described in section IV for sufficiently large α , $L(q)$ should cross zero leading to power law tails.

The final example considered here is the population growth model, eqn. (3), with $\alpha = 0$. This has been studied extensively because of its relevance to quantum mechanical many body systems. The Lyapunov exponent $L(q)$ corresponds to the ground state energy of q particles interacting with each other via a two body potential as was first shown by Sugiyama and Koonin [18]. Here we are considering a lattice model so that \mathbf{r} takes discrete values and ∇^2 is the discrete Laplacian. Without the last term, the above equation is just the diffusion equation, or the imaginary time Schroedinger equation for a free particle. The inter-particle interaction is a result of the multiplicative term as we shall see shortly. Assume that f is Gaussian noise with zero mean and the correlation function $\langle f(\mathbf{r}, t) f(\mathbf{r}', t') \rangle = v(\mathbf{r} - \mathbf{r}') \delta(t - t')$.

For discrete \mathbf{r} consider the n th moment averaged at equal times

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n, t) \equiv \langle \phi(\mathbf{r}_1, t) \phi(\mathbf{r}_2, t) \dots \phi(\mathbf{r}_n, t) \rangle. \quad (18)$$

In the notation of section III, $\Psi = \vec{\phi}_{\vec{\alpha}}$ and $\vec{\alpha}$ corresponds to the coordinates $(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n)$. Translating eqns. (8-10) into these new symbols it can be seen that Ψ obeys the many body Schroedinger equation with imaginary time.

$$-\dot{\Psi} = H\Psi \quad (19)$$

$H = K + U$ is the Hamiltonian of the system which is the sum of the kinetic and potential energy, K and U respectively

$$K = - \sum_{i=1}^n \nabla^2 \quad (20)$$

and

$$U = - \sum_{i < j}^n u(\mathbf{r}_i - \mathbf{r}_j) - \frac{n}{2} u(0) \quad (21)$$

Note that u need not be positive as f may be chosen to be complex. This allows for the simulation of systems of particles with both attractive and repulsive interactions. Also note that the last term in the potential energy is constant and therefore does not present

any difficulties. The addition of a term $-U_{ext}(\mathbf{r})\phi(\mathbf{r})$ to eqn. (3) incorporates an external potential U_{ext} in the Hamiltonian H .

Thus the wavefunction of a system of n bosons, with certain initial conditions, is simply related to the n th moment of equation (3). To obtain the ground state energy of n bosons, E_n , one need calculate the Lyapunov exponent $L(n) = -E_n$. One is normally not interested in the constant term $-\frac{n}{2}u(0)$ that appears in the expression for the potential energy, so this is subtracted out. Eqn. (3) has been studied numerically by path integral techniques [18]. It is worth noting that the replication algorithm presented here is also an efficient procedure for obtaining ground state energies. Again, straightforward simulation of this equation will not work because the higher order moments will not converge in a reasonable time. The replication algorithm efficiently solves this problem.

The method with the addition of the replication algorithm can handle large number of particles and is quite computationally efficient. Fig. 6 shows the difference in energies $\mu_n \equiv E_n - E_{n+1}$ as a function of n for a 10×10 system with an on-site attractive potential $U = -0.5$. Approximately 900 copies are run in this simulation. In two dimensions there is a transition between localized and delocalized particles as a function of density. The two dimensional transition can be understood as a competition between kinetic and potential energy. The kinetic energy scales as n/R^2 where R is the correlation length. The potential energy scales as $un(n-1)^2/(2\pi R^2)$. In the delocalized phase R is the system size L , and in the localized phase R should become as small as possible which is one lattice spacing. This corresponds to all particles occupying the same lattice site. The transition between localized and delocalized phases will then occur at approximately $n = 2\pi/u$. For the figure shown the transition occurs within a factor of two from the simple estimate given above. This transition is not a thermodynamic transition as it occurs at a constant number of particles, almost independent of volume.

An interesting feature of the replication algorithm applied to this system is that it exhibits metastability and hysteresis right above the transition. If the initial field $\phi(\mathbf{r})$ is delocalized then it will stay so for many iterations of the replication algorithm. It will eventually jump to the localized phase and remain in that state. The interpretation of this is similar to nucleation theory [32].

VI. CONCLUSIONS

The PDF for eqn. (1) has been analyzed for many components. It was found that PDF will exhibit power law tails when a symmetry condition holds for the random multiplicative matrix \mathbf{M} .

A passive scalar such as temperature in a random velocity field is predicted to have a PDF with an exponential or stretched exponential tail in agreement with earlier experimental and theoretical work [2,5–11]. If the fluid is reactive this is predicted to become power law when the reaction rate is sufficiently large. Numerical evidence was presented supporting this prediction. The numerical work suggests that this power law behavior is most easily seen in small systems where the velocity correlation length is comparable to system size.

It is interesting that the multi-component Gaussian case should exhibit the same behavior as the a one component system albeit one that is no longer gaussian. An argument can be given to understand this equivalence with no additive noise. A multidimensional system has a variety of relaxation times. Consider the longest of these T_{rel} . For times much longer than T_{rel} a system looks statistically identical to how it did originally except for an overall shift in scale. To make this concrete, suppose we shift the scale periodically after a time $\Delta T \gg T_{rel}$ so that $|\vec{\phi}|^2 = 1$. The shift in scale will not be completely deterministic but will fluctuate because of the stochastic nature of the equation. For a long time $M\Delta T$, the overall scale factor is obtained by multiplying the M shifts in scale together. Each of these scale shifts is random and in general is expected to be non-gaussian. The nature of this non-gaussian distribution depends on the evolution of the system within a relaxation time. Therefore the long time evolution of this system can still be properly modeled by a one component system as was investigated previously [25]

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FIGURES

FIG. 1. The probability distribution of a field ϕ obeying eqn. (15) in the absence of additive noise. The system is in one dimension and is plotted with rescaled axis. The horizontal axis is $\ln \phi/t$ and the vertical axis is $(\ln P)/t$. Two different times $t_1 = 12.8$ and $t_2 = 38.4$ are shown. The variance of the multiplicative noise is 2.0, the diffusion coefficient d is 0.1.

FIG. 2. The probability distribution of field ϕ obeying eqn. (14) in one dimension, Six different values of α are shown. The most steeply descending curve is for $\alpha = 0$, and in order of decreasing steepness, $\alpha = 0.40, \alpha = 0.47, \alpha = 0.49, \alpha = 0.50$, with the top curve at $\alpha = 0.51$. The equation was discretized and the number of lattice sites was chosen to be 8. The variance of the both the multiplicative and additive noise is 0.6.

FIG. 3. The logarithm of $\langle \phi \rangle$ as a function of time, for a two dimensional passive scalar field. The average is a weighted average corresponding to the weights ϕ^1 and ϕ^8 and ϕ^{16} , calculated using the replication algorithm described in the text. The slopes of these lines correspond to $\Delta L \equiv L(q+1) - L(q)$, with $q = 1, 8$ and 16 . The bottom curve corresponds to $q = 1$, the middle curve to $q = 8$, and the top curve is for $q = 16$.

FIG. 4. The difference in Lyapunov exponent ΔL as a function of q for the two dimensional motion of a passive scalar field on an 8×8 lattice. Note all of the values are monotonically decreasing confirming that $L(q)$ is convex.

FIG. 5. The difference in Lyapunov exponent ΔL as a function of q for the same system as the previous figure but on a 16×16 lattice. Note the variation of $L(q)$ with q is much smaller in this larger system.

FIG. 6. The chemical potential at zero temperature, $\mu = E_n - E_{n+1}$ as a function of n , the number of bosons in a two dimensional 10×10 lattice. The on-site attractive energy is -0.5 . The energies are calculated by the replication method described in the text. Note the discontinuity at $n = 19$.

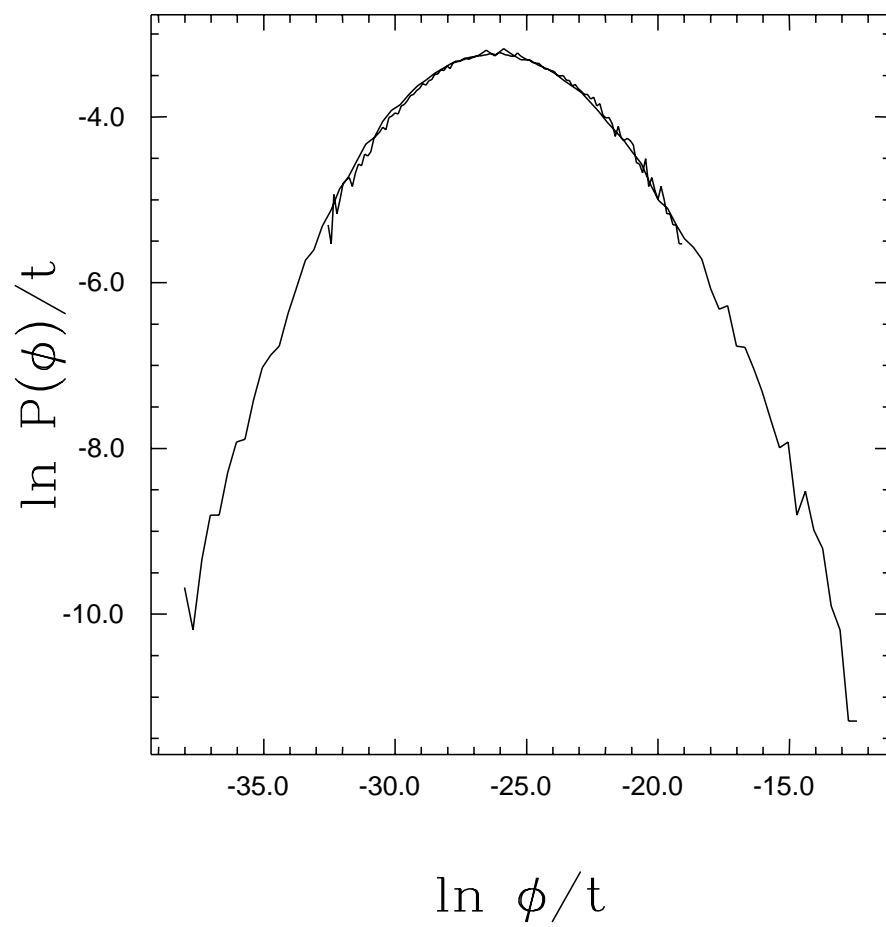


Fig. 1

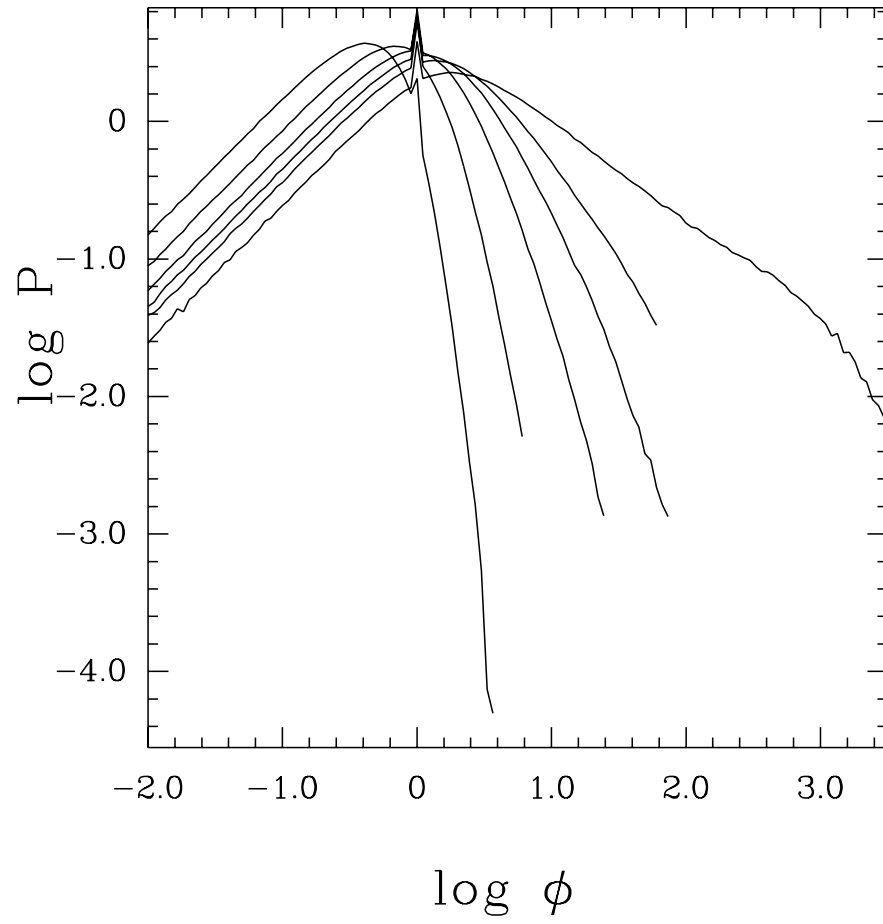


Fig. 2

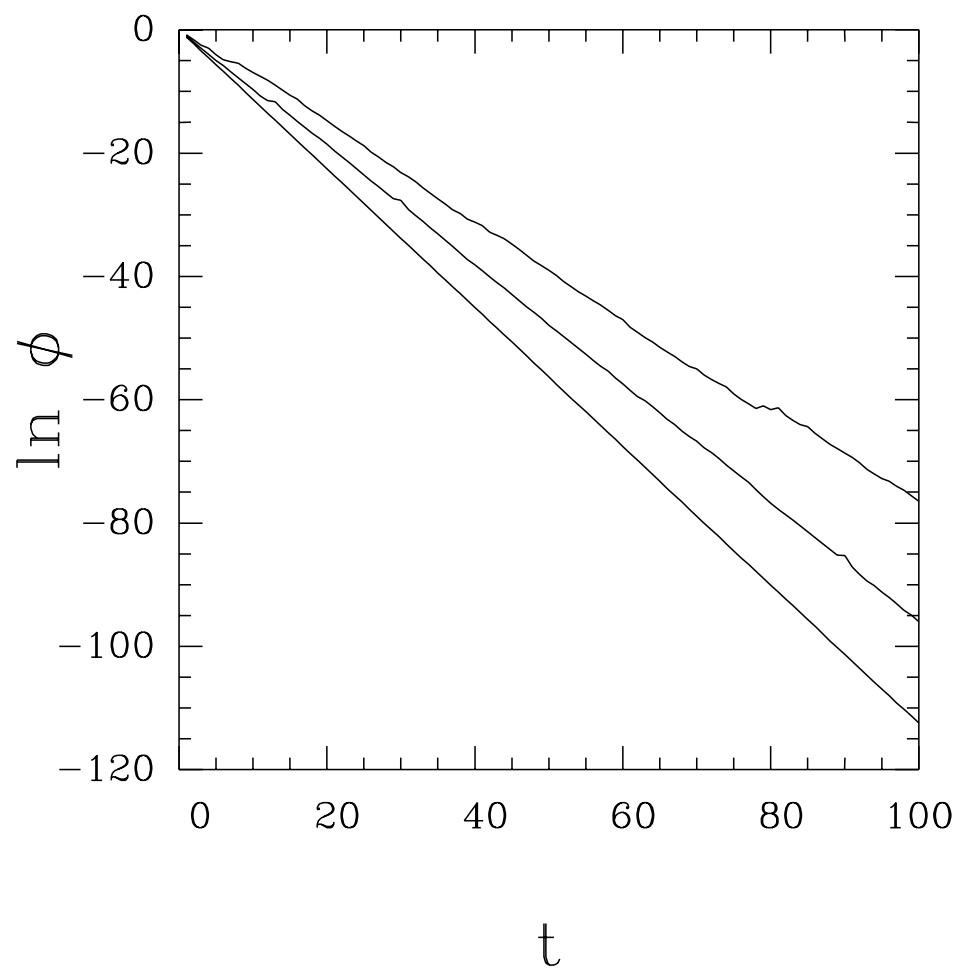


Fig. 3

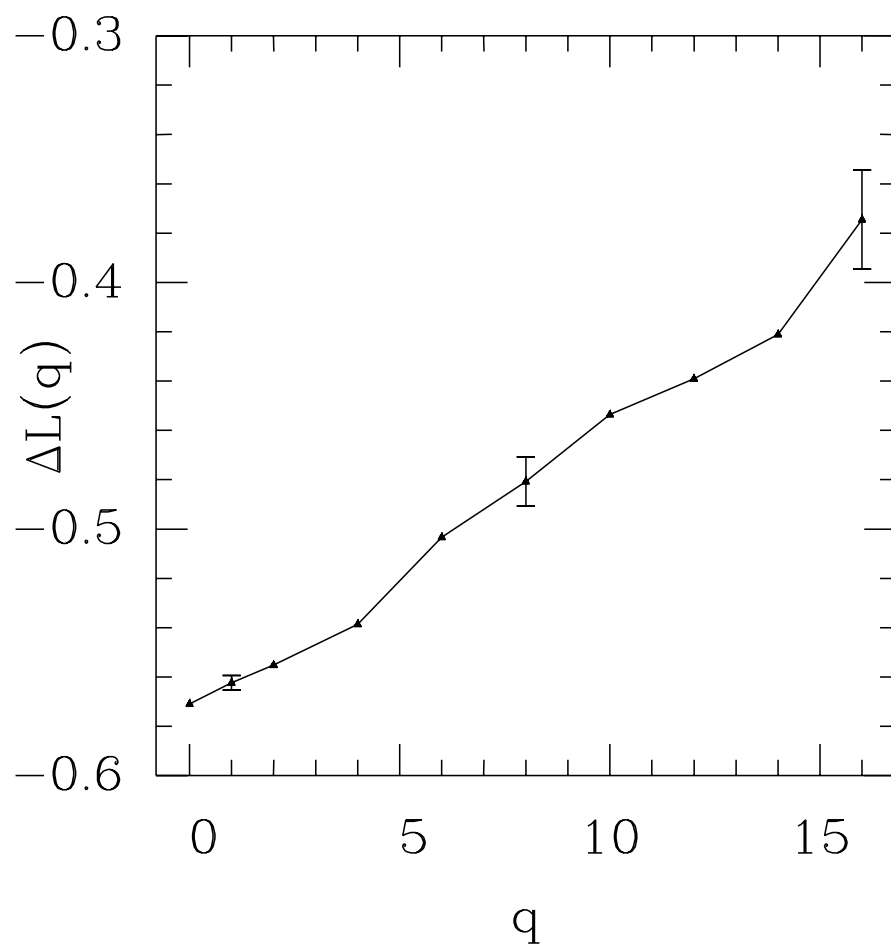


Fig. 4

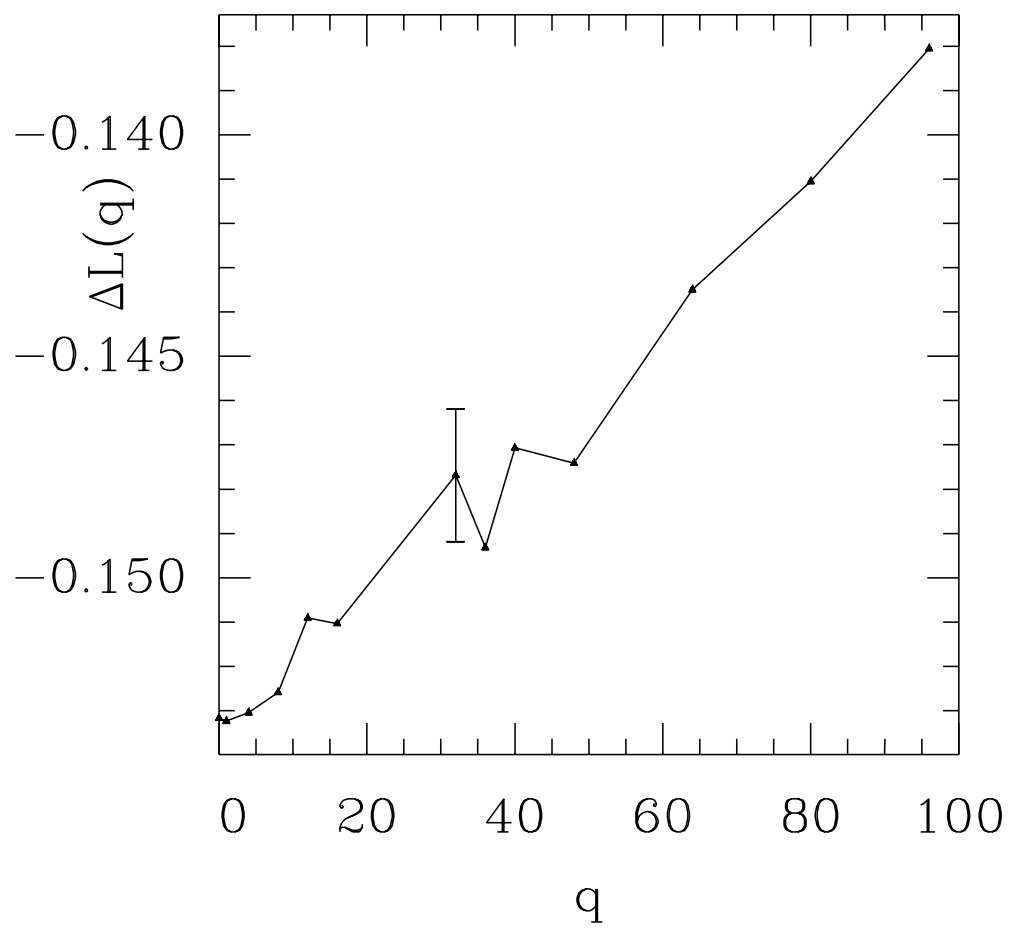


Fig. 5

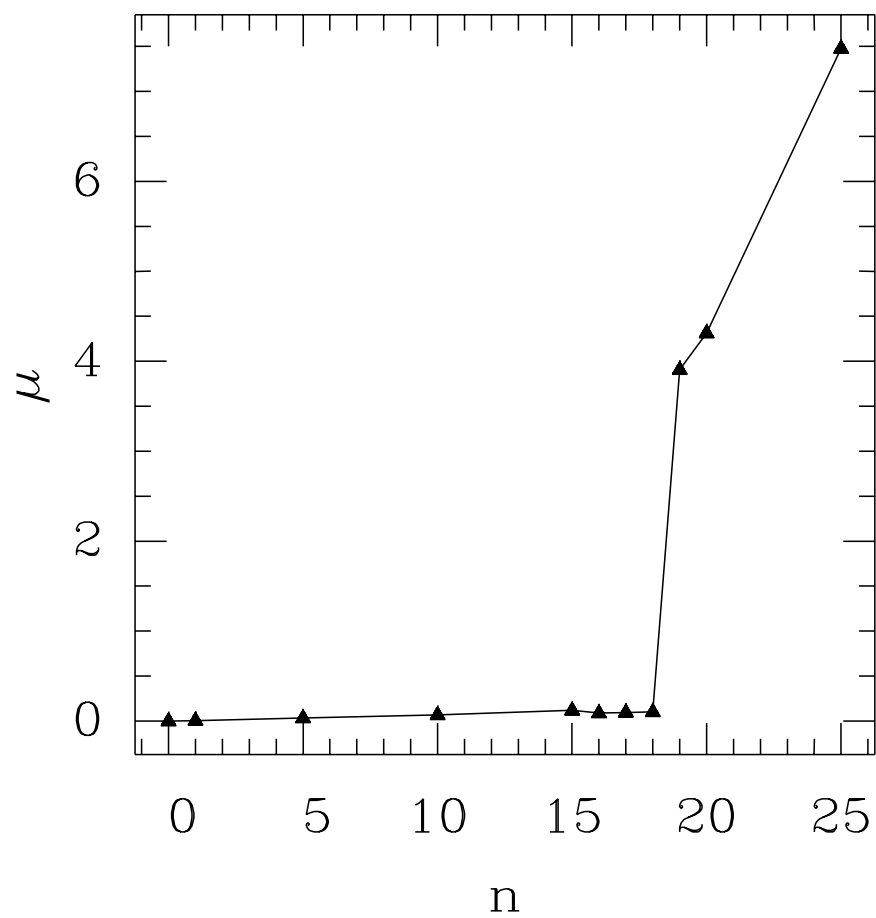


Fig. 6